# Orthogonal Polynomial Method & Double Scaling Limit ①

~~remark~~

$\boxed{Q}$ where does the $\underbrace{\text{singular behavior of } g}_{\text{fractional power}}$

come from?

essential for to have the double-scaling limit.

$\boxed{A}$ " $\lambda \sim 2\sqrt{a(g)}$ " determines the singular behavior.
~~to answer the question, it's useful to consider~~

the orthogonal poly. method is extremely useful!

to show this

§ orthogonal polynomial

(i)

$$\int_{-\infty}^{\infty} d\lambda\, e^{-NV(\lambda)}\, P_n(\lambda) P_m(\lambda) = h_n \delta_{nm}$$

definition

$P_n(\lambda)$: n-th order polynomial

e.g. $V(\lambda) = \frac{1}{2}\lambda^2$ then, $P_n(\lambda)$ becomes Hermite polynomial.

$$e^{-NV(\lambda)/2} P_n(\lambda) \propto \psi_n(\lambda)$$

~~wavefunction~~ energy eigenstate of harmonic oscillator

(ii) recursion relation.

depends on "$g$"

$$\underline{\lambda P_n(\lambda) = P_{n+1}(\lambda) + S_n P_n(\lambda) + R_n P_{n-1}(\lambda)}$$

$S_n = 0$ for even $V(\lambda)$ (for this case, one can show that $P_n(-\lambda) = (-1)^n P_n(\lambda)$)

Q no other terms such as $P_{n-2}(\lambda)$ ?

$$\int_{-\infty}^{\infty} d\lambda \, e^{-NV(\lambda)} \lambda P_n(\lambda) P_{n-2}(\lambda) = \int_{-\infty}^{\infty} d\lambda \, e^{-NV(\lambda)} P_n(\lambda) \Big( P_{n-1}(\lambda) + \text{polynomial of lower deg} \Big)$$

$$= 0 \text{ ! identically}$$

$\therefore$ $\lambda P_n(\lambda)$ does not contain $P_{n-2}(\lambda) \cdots P_0(\lambda)$

(iii) two relations that $R_n(\lambda)$ satisfy.

① $$\int d\lambda \, e^{-NV(\lambda)} \lambda P_{n-1}(\lambda) P_n(\lambda) = h_n(g)$$

$$\parallel$$

$$h_{n-1}(g) \cdot R_n(g)$$

Hence $R_n = h_n / h_{n-1}$

② $$\underbrace{\int d\lambda\ e^{-NV(\lambda)}\ \frac{d}{d\lambda} P_n}\cdot P_{n-1} = n\, h_{n-1}$$

$$= N \int d\lambda\ \overset{e^{-NV(\lambda)}}{(\lambda + 4g\lambda^3)} \cdot P_n \cdot P_{n-1}$$

$$= N R_n h_{n-1} + 4gN \int \overset{e^{-NV(\lambda)}}{\left( \cdots + R_{n+1} R_n P_{n-1} + R_n^2 P_{n-1} + R_n R_{n-1} P_{n-1} \right)} P_n$$

$$= N \cdot R_n \left( 1 + 4g \left( R_{n+1} + R_n + R_{n-1} \right) \right) h_{n-1}$$

$$\therefore \boxed{\frac{n}{N} = R_n \left( 1 + 4g \left( R_{n+1} + R_n + R_{n-1} \right) \right)}$$

(iv) partition function

$$Z[g] = \int d\lambda_1 \cdots d\lambda_N\ \underbrace{\Delta^2(\lambda_i)}\ e^{-N(V(\lambda_1) + \cdots V(\lambda_N))}$$

by def

$$= \det{}^2 \left( \lambda_i^{j-1} \right)$$

$$= \det{}^2 \left( P_{j-1}(\lambda_i) \right)$$

$$= \left[ \varepsilon^{j_1 \cdots j_N} P_{j_1}(\lambda_1) P_{j_2}(\lambda_2) \cdots P_{j_N}(\lambda_N) \right]^2$$

one can obtain that

$$Z[g] = N!\, h_0 h_1 \cdots h_{N-1}(g)$$

$$Z[g] / Z[0] = \left(\frac{h_0(g)}{h_0(0)}\right)^N \left(\frac{R_1(g)}{R_1(0)}\right)^{N-1} \left(\frac{R_2(g)}{R_2(0)}\right)^{N-2} \cdots \left(\frac{R_{N-1}(g)}{R_{N-1}(0)}\right)$$

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## § large N limit

(i) recursion relation in the $N \to \infty$ limit

$$\frac{n}{N} \equiv x. \qquad r(x = \frac{n}{N}, g) \equiv R_n(g)$$

then, the recursion relation can be written as

$$x = r(x,g)\Big(1 + 4g\,\underbrace{(r(x+\varepsilon, g) + r(x,g) + r(x-\varepsilon, g))}_{\text{leading term } \simeq\, 3r_0(x,g) \text{ (sphere) contribution}}\Big)$$

$$\therefore \quad r_0 \cdot (1 + 12 g r_0) = x \quad \text{or} \quad r_0(x,g) = \frac{-1 + \sqrt{1 + 48 g \cdot x}}{24 g}$$

$$r_0(x = 1, g) = a^2(g)$$ ← end-point of eigenv

(ii) partition function.

$$\log\left(Z[g]/Z[0]\right) = N\log\left[h_0(g)/h_0(0)\right]$$

$$+ N\cdot\sum_{n=1}^{N-1}\left(1-\frac{n}{N}\right)\log\left(R_n(g)/R_n(0)\right)$$

N→∞ limit

$$\cong N^2\underbrace{\int_0^1 dx\,(1-x)\log\left(r_0(x,g)/\underset{x}{\underset{\|}{r_0(x,0)}}\right)}$$

$$= -E^{h=0}(g). \qquad (\underline{\text{sphere}})$$

when $r(x,g)\cong r_0(x,g)$.

one can explicitly show that

$$\int_0^1 dx\,(1-x)\log\left(r_0(x,g)/x\right) \quad \text{where} \quad r_0(x,g)=\frac{-1+\sqrt{1+48gx}}{24g}$$

$$= \frac{1}{2}\log a^2(g) - \frac{1}{24}\left(9-a^2(g)\right)\left(a^2(g)-1\right) \quad \square$$

$\boxed{\text{§ singular behavior}}$

(i) notice that the scaling behavior we want is



$$+E^{h=0}(g) \sim (g_c - g)^{\frac{5}{2}}$$

(ii)
$$-E^{h=0}(g) = \int_0^1 dx\,(1-x)\log r_0(x,g)$$

assume that
$$\boxed{\begin{array}{c} r_0(x,g) \underset{g\to g_c}{\cong} r_c + (-g_c + gx)^{\frac{1}{2}} \\ \text{(and near } x=1\text{)}\end{array}}$$

$$\underset{g\to g_c}{\cong} \int_0^1 dx\,(1-x)\log\underbrace{(r_c + (-g_c+gx)^{\frac{1}{2}})}_{\cong (-g_c+gx)^{\frac{1}{2}}} + \underbrace{(\cdots)}_{\text{irrelevant}}$$

$$\sim \underbrace{+(1-x)(-g_c+gx)^{\frac{3}{2}}\Big]_0^1}_{\text{irrelevant}} + \underbrace{\int_0^1 dx\,(-g_c+gx)^{\frac{3}{2}}}_{\sim (-g_c+g)^{\frac{5}{2}} + (\cdots)}$$

(※ irrelevant terms are of order "1")
they become divergent in $N\to\infty$. (cut-off ?? )

$\underline{\underline{(-g_c+g)^{\frac{5}{2}}}}$ !! this is what we want !!

(iii) Can we obtain this behavior ?

$$x = W(r)$$

$$= r_0 \left( 1 + 12 g r_0 \right)$$

$$= -\frac{1}{48g} + 12g \left( r_0 + \frac{1}{24g} \right)^2$$

or $$\underbrace{gx + \frac{1}{48}}_{= -g_c} = 12 g^2 \left( r_0 + \frac{1}{24g} \right)^2$$

$$\Leftrightarrow \quad r_0(x) + \frac{1}{24g} = \cancel{\sqrt{\frac{1}{12} g (gx - g_c)}} \sqrt{\frac{1}{12g^2}} \, (gx - g_c)^{\frac{1}{2}}$$

this is exactly what we want

(iv) in general,

$$x = W(r)$$

$$\cong W(r_c) + \frac{1}{2} W''(r_c)(r - r_c)^2 + \underbrace{(\cdots)}$$

one need to argue that
these terms become
negligible when $x \sim 1$

Here $r_c$ is defined as a critical point, i.e.,

$$W'(r = r_c) = 0 .$$

to make everything consistent, one has to require

$$g x \underset{(x \sim 1)}{\cong} \boxed{\underbrace{g\, W(r_c)}_{= g_c}} + \frac{1}{2} W''(r_c) \cdot g \cdot (r - r_c)^2 + \cdots$$

# § all genus partition function



note that

$$\log\left(Z_N[g] / Z_N[0]\right) \underset{N\to\infty}{\simeq} N^2 \int_0^1 dx\,(1-x)\,\log[r(x,g)/x]$$

where

$$x = r(x)\left(1 + 4g\left(r(x+\tfrac{1}{N}) + r(x-\tfrac{1}{N}) + r(x)\right)\right)$$

Using the double-scaling limit, one can actually obtain the exact parratition function in terms of "$\hbar$"!

※ double-scaling limit

$$\left\{\begin{aligned} g &= g_c + \alpha^4 \kappa \qquad && g_c = -\frac{1}{48}\ \ \&\ \ 48g = -1 + 48\kappa\alpha^4 \\ \frac{1}{N} &= \hbar\alpha^5 && \\ r(x) &\cong r_c - 2\alpha^2 u(y) && \text{where}\quad x = \underbrace{\phi}_{W(r_c)} - \alpha^4 y \end{aligned}\right\}$$

$$-\frac{1}{24g} \cong 2 + \#\alpha^4 \quad \text{negligible}$$

$$W(r_c) = -\frac{1}{48g} \cong 1 + 48\alpha^4\kappa$$

then, one can argue that

$$\log\left(Z_N[g]/Z[0]\right) \underset{\substack{N\to\infty \\ \alpha\to 0}}{\sim} \left(\hbar^{-2}\alpha^{-10}\right)\int_{\infty}^{48\kappa} (-\alpha^4 dy)(\alpha^4(y-48\kappa))\underbrace{\log(r_c-2\alpha^2 u(y))}_{\simeq -\alpha^2 u(y)}$$

$$= \underbrace{-\hbar^{-2}\int_{48\kappa}^{\infty} dy\,(y-48\kappa)\,u(y,\hbar)}_{\text{"exact" } Z_{\text{gravity}} \text{ in terms of } \hbar\,!} \qquad \star$$

where $u(y,\hbar)$ satisfies the eqn. below

$$y = \underset{\text{non-linear}}{u^2(y)} - \frac{\hbar^2}{3}\frac{d^2}{dy^2}u(y) \qquad \underline{\underline{\text{Painlevé I}}}$$

(i) the above eqn. can be obtained by taking the <u>double-scaling limit</u> of the relation below

$$\underset{1-\alpha^4(y-48\kappa)}{x} = \underbrace{r(x)\left(1+12g\,r(x)\right)}_{\substack{=W(r_c-2\alpha^2u(y)) \\ \cong 1+48\kappa\alpha^4-\alpha^4u^2(y)}} + 4g\,r(x)\cdot\underbrace{\left(r(x+\hbar\alpha^5)+r(x-\hbar\alpha^5)-2r(x)\right)}_{\sim\alpha^4\frac{1}{3}\frac{d^2}{dy^2}u(y)}$$

(ii) one can try to solve the Painlevé I eqn

perturbatively in $\hbar$

$$u(y) = \sqrt{y} + \hbar^2 w(y)$$

$$\rightarrow \cancel{y} = (\cancel{y} + 2\hbar^2 \sqrt{y}\, w(y)) - \frac{\hbar^2}{3}\left(-\frac{1}{4}\right) y^{-\frac{3}{2}} + \cdots$$

$$\therefore w(y) = -\frac{1}{24} y^{-2} + O(\hbar^2)$$

one can argue that

$$u(y) = \sqrt{y}\left(1 - \sum_{k=1}' u_k \hbar^{2k} \cdot (y^{-\frac{5}{2}})^k\right)$$

plugging them into eq. ⊛ leads to all-genus result.

all positive.

※ remark

this series is a asymptotic series & non-Borel summable

$\Rightarrow$ non-perturbative in $\hbar$ i.e $\exists$ $e^{-\frac{1}{\hbar}}$ factor in the solution $u(y)$ $\longrightarrow$ $\boxed{\text{physical meaning}}$ ??